THE PRIMAL AND DUAL METHODS OF SENSITIVITY ANALYSIS OF THE SOLUTIONS OF GENERAL OPTIMIZATION PROBLEMS, WITH A COMPARISON OF THEIR RESPECTIVE STRENGTHS AND WEAKNESSES

by

Emmanuel Drandakis
Professor of Economics, Emeritus
Athens University of Economics and Business

Abstract

The paper examines parametric equality-constrained optimization problems in matrix theory terms and compares the primal and dual methods developed for sensitivity analysis of parametric variation.

The primal method uses **Caratheodory's** (1935) theorem on the properties of the inverse of the bordered hessian. When parameters appear only in the objective function, sensitivity analysis proceeds almost as smoothly as in unconstrained optimization problems.

However, when parameters appear in all functions, **Caratheodory's** theorem leads to s-o-conditions on the hessian of the problem that are subject to constraints and, thus, cannot be used as such for sensitivity analysis. The difficulty was faced by **Silberberg** (1974); yet, the impact of Caratheodory's theorem was not completely destroyed but only limited, as it was restricted to the hessian's tangent subspace.

As **Luenberger** (1973) has shown, a matrix restricted on the tangent subspace can easily be specified and that reduced matrix must indeed be negative semi-definite or definite. Thus, the primal method is still usable in a subspace of the space of choice variables.

As for the dual method, we first note that it cannot work when parameters enter the constraint functions, if the number of parameters does not exceed that of constraints. Otherwise, the Envelope theorem provides the appropriate curvature conditions on the tangent subspace of the hessian of the dual problem. These envelope curvature conditions can be easily specified, using again **Luenberger**'s method, to produce a reduced matrix which is negative semi-definite or definite on the tangent subspace of the hessian of the dual problem.

As the number of parameters increases beyond that of choice variables, the dual method should be preferred to the primal, since the tangent subspace of the dual becomes greater than that of the primal.

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1. Introduction

In a series of papers, **Drandakis** (2003), (2007), (2009) and (2010) has examined the Primal and Dual methods for sensitivity analysis in general parametric equality – constrained optimization problems. Today such analysis is conducted in matrix theory terms pioneered by "econometricians", like **Barten**, **Dhrymes** and others in the late 1960's. The Primal method of sensitivity analysis uses **Caratheodory's** (1935) theorem on the properties of the Inverse of the bordered hessian of the optimization problem, that are needed for determining the signs of the rates of change of the choice variables as parameters vary, by deriving the fundamental matrix equation of sensitivity analysis. The Dual method was introduced a little bit later by **Silberberg** (1974) and **Hatta** (1980), who examined dual optimization problems, in which parameters assume the role of choice variables and vise versa and, in effect, consider appropriate Envelope theorems for deriving Envelope tangencies and Envelope curvature conditions suitable for sensitivity analysis.

The present paper compares the Primal and Dual methods of sensitivity analysis in general constrained optimization problems, when parameters appear in the objective and/or constraint functions of the problem. Sensitivity results do depend on the above distinction and also, on whether parameters enter the constraints only linearly, as constraint levels, or not. When parameters enter all functions, then second-order conditions for a local maximum involve the negative semi-definiteness (or definiteness) of the hessian matrix, only when that matrix is restricted on its tangent subspace. In all such cases, to produce curvature conditions suitable for sensitivity analysis, we have to derive the reduced matrix that represents the hessian on its tangent subspace, as **Luenberger** (1973) has shown. Thus sensitivity results are derived for an n – r subspace of choice variables, where r is the number of constraints.

On the other hand, the dual method is based on a dual problem in which the former choice variables are treated as fixed parameters which parameters are the choice variables. Both **Silberberg** (1974) and **Hatta** (1980) examine the difference between the objective function at a fixed $x^o = x(\alpha^o)$, and the maximal value function of the primal model, $f(x^o, \alpha) - \varphi(\alpha)$, which has a local maximum of zero for feasible values of α, $h(x^o, \alpha) = 0_r$ or for any α in an open subset of R^m, where $h(x^o) = 0$, or $h(x^o) - \gamma = 0_r$. It is evident that the maximum of zero in all such cases is nothing else but an application of a constrained or an unconstrained Envelope theorem, leading to Envelope tangencies and Envelope curvature conditions which may require the representation of the hessian matrix of the dual problem as it is restricted on its tangent subspace, if the Envelope problem is a constrained one. In that case we must specify a matrix representing the hessian restricted on its tangent subspace, evaluate this reduced matrix, using the second-order derivative properties of the maximal value function of the original problem and show it as simply as possible, so that the meaning of its semi-definiteness (or definiteness) become transparent. As for **Hatta** (1980), who examined a less general but easier primal and dual problems, in which some parameters appear linearly in the constraints as no-zero constraint levels, he was able to derive an unconstrained Envelope problem leading directly to a semi-definite (or definite) matrix of the rates of change of the optimal solution.

The paper is organized as follows. *Section 2* examines an equality – constrained optimization problem, in which parameters are constant and do not appear explicitly, and considers the main features of **Caratheorody's** (1935) theorem on the properties of the Inverse of the bordered hessian matrix of the problem. *Section 3* examines parametric variations and the Primal method of sensitivity of the rates of change of choice variables. Depending on the parametric structure of the problem, we examine separately sensitivity results

using the **Caratheodory** theorem. When parameters appear only in the objective function of the problem, sensitivity analysis runs almost as smoothly as it does in unconstrained optimization problems. Difficulties begin when parameters appear in all functions. The s-o-conditions then specify that the hessian of the problem must be negative semi-definite (or definite) **only when** it is restricted on its tangent subspace. To be able to proceed with sensitivity analysis, we must derive the restricted hessian on that subspace. In the general problem examined by **Silberberg** (1974), we derive an (n – r) x (n – r) restricted hessian and thus get sensitivity results for (n – r) choice variables. In the simpler problem examined by **Hatta** (1980), in which r parameters appear in the constraints linearly, as constraint levels, the difficulty is overcome, but with other parameters appearing non-linearly in the constraint functions, we get again sensitivity results for n – r choice variables. *Section 4* examines the Dual method of sensitivity analysis developed in the two basic contributions of **Silberberg** (1974) and **Hatta** (1980). Both papers consider a dual optimization problem, in which parameters behave as choice variables and vice versa. The Envelope theorem is used in comparing the value of the objective function at a fixed $x^o = x(\alpha^o)$ to that of the maximal value function of the original problem.

At the local maximum of zero of the constrained Envelope problem, of **Silberberg** (1974), envelope tangencies emerge along with second-order conditions that must hold under constraints, i.e., on the tangent subspace. Then, s-o-conditions have to be specified as they appear for the restricted Hessian of the dual problem on its tangent subspace. Again we follow **Luenberger**'**s** (1973) method and produce sensitivity results for m - r choice variables of the original problem, where m = number of parameters. Finally, *Section 5* concludes with a review of the development of both the Primal and the Dual methods of sensitivity analysis and a comparison of their respective strengths and weaknesses.

Our analysis relies on classical optimization techniques in matrix theory terms. All vectors are treated as column vectors, unless they are enclosed within parentheses or appear as function arguments, while matrices are denoted by capital letters. Then e.g. 0, 0_m, or O_{mn} indicate the zero scolar, a vector of m zeros, or an m x n matrix of zeros, respectively, while $x_i(\alpha) \in R$ and $x(\alpha) \in R^n$ need no explanation but $X_\alpha(\alpha)$ denote the n x m matrix of the partial derivatives of $x_i(\alpha)$ in α_j, $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Finally a prime offer a vector or a matrix denotes transposition.

2. The Caratheodory (1935) theorem

We examine first the equality-constrained optimization problem

$$\max_x \{f(x) \mid h(x)' = 0_r'\} , \qquad (P)$$

with n choice variables $x \in X$, an open subset of R^n, r constraints. $1 \leq r < n$, and parameters which are initially constant and do not appear explicitly in (P).

$x^o \in X$ is a feasible point of (P), if $h(x^o) = 0_r$, and is a **regular** feasible point, if the rank of the gradient matrix of $(h_1(x), \ldots, h_r(x))$ is equal to r, i.e.,

$$\rho(H_x(x^o)) = r . \qquad (R_x)$$

If x^o is regular and a local maximum of (P), then there exist r lagrangena multiplier $\lambda^{o'} = (\lambda_1^o, \ldots, \lambda_r^o)$ and the necessary first-order

$$\text{f-o-c} \quad \{f_x(x^o) - H_x(x^o)\lambda^o = 0_n , h(x^o) = 0_r\} \qquad (1)$$

and second - order – conditions

$$\text{s-o-c}\left\{\begin{array}{l}\text{The hessian matrix of (P) is negative}\\ \text{semi}-\text{definite on the tangent}\\ \text{subspace } T_{x^o}=\{\eta\in R^n \;\big|\; H_x(x^o)'\,\eta=0_r\}\end{array}\right\} \qquad (2)$$

are satisfied. On the other hand if x^o, λ^o satisfy (1) and (2_s), or

$$\text{s-o-s-c}\left\{\begin{array}{l}\text{The hessian matrix of (P)}\\ \text{is negative definite on the tangent}\\ \text{subspace } T_{x^o} \text{ and } \eta\neq 0_n\end{array}\right\} \quad , \qquad (2_s)$$

then x^o attains a strict local maximum of (P), the Implicit – function – theorem works and x^o and λ^o are both continuously differentiable functions of parameters.

Finally, if x^o and λ^o satisfy (R_x), (1) and (2_s), it is well known that the bordered hessian of (P) is an invertible (n + r) x (n + r) matrix.[1]

We note the difference between the n x r gradient matrix of $h(x^o)$, or $H_x(x^o)$, and the hessian of (P), or the n x n matrix $A = A(x^o, \lambda^o) = F_{xx}(x^o) - H_{xx}(x^o, \lambda^o)$, where $H_{xx}(x^o,\lambda)^o \equiv \sum_{j=1}^{r}\lambda_j^o H_{xx}^j(x^o)$ is the sum of the matrices of second-order derivatives of all $h_j(\alpha_1)$ in x weighed by their respective lagrangean multiplier. Finally, let $B = H_x(x^o)$ and the bordered hessian of (P), or $\begin{bmatrix} A, & B \\ B', & O_{rr}\end{bmatrix}$, whose Inverse matrix $\begin{bmatrix} U, & V \\ V', & W\end{bmatrix}$ exists at x^o, λ^o.

Caratheodory's (1935) theorem provides all the needed information about V and V′ whose rank is equal to r, about U, which is an n x n symmetric and negative semi-definite matrix, with $\rho(U) = n - r$, and about the r x r

[1] For a proof in matrix terms see e.g. **Drandakis** (2003) lemma 1.

symmetric matrix W, which is positive (semi)-definite when A happens to be negative (semi) definite since $V'A\,V = -\,W$. [2]

3. Parameters may vary. The Primal method of Sensitivity analysis

(i) Parameters appear in the objective function and may vary

We consider explicitly m parameters $1 < m$, that appear in the objective function of (P) and examine the effects of their variation on the rates of change of the solution of (P).

Let the behavioral system be

$$\phi(\alpha) \equiv \max_{x} \{ f(x,\alpha) \;\big|\; h(x)' = 0_r' \} \;, \qquad (P^{(i)})$$

with m parameters $\alpha \in A$, an open set of R^m. When a vary, sensitivity analysis proceeds almost as smoothly, via the **Caratheodory** theorem, as it would in the case of an unconstrained optimization problem. This is rather surprising, if we take into account that the s-o-s-conditions of $(P^{(i)})$ are subject to constraints and, thus, cannot be used as they stand for sensitivity analysis.

However, since $x(\alpha)\,,\ \lambda(\alpha) \in C'^1$ functions of α near $\alpha^o \in A$, we can consider the bordered hessian of $(P^{(i)})$, or $\begin{bmatrix} A\,, & B \\ B', & O_{rr} \end{bmatrix}$ and its inverse $\begin{bmatrix} U\,, & V \\ V', & W \end{bmatrix}$.

Indeed if we differentiate the f-o-identities

$$\{ f_x(x(\alpha^o),\alpha^o) - H_x(x^o)\lambda(\alpha^o) = 0_n \;,\; h(x(\alpha^o)) = 0_r \} \qquad (3)$$

[2] The above and other results are proved quite easily in matrix terms, without requiring the compotation of the Inverse. See e.g. **Drandakis** (2003).

w/v to x and (-λ), the **Jacobian** matrix $\begin{bmatrix} F^o_{xx} - H^o_{xx} & , & H^o_x \\ H^{o'}_x & , & O_{rr} \end{bmatrix} = \begin{bmatrix} A & , & B \\ B' & , & O_{rr} \end{bmatrix}$ is non other but the bordered hessian of ($P^{(i)}$) and the fundamental matrix equation for sensitivity analysis is

$$\begin{bmatrix} A , & B \\ B' , & O_{rr} \end{bmatrix} \begin{bmatrix} X_\alpha(\alpha^o) \\ -\Lambda_\alpha(\alpha^o) \end{bmatrix} = \begin{bmatrix} -F_{x^o\alpha} \\ O_{rr} \end{bmatrix}, \text{ or}$$

$$\begin{bmatrix} X_\alpha(\alpha^o) \\ -\Lambda_\alpha(\alpha^o) \end{bmatrix} = \begin{bmatrix} U , & V \\ V' , & W \end{bmatrix} \begin{bmatrix} -F_{x^o\alpha} \\ O_{rr} \end{bmatrix} = \begin{bmatrix} -UF_{x^o\alpha} \\ -V'F_{x^o\alpha} \end{bmatrix} . \qquad (4)$$

We see then that the m x m matrix

$F^o_{\alpha x}\ X_\alpha(\alpha^o) = -F^o_{\alpha x}\ UF^o_{x\alpha}$ is symmetric and positive semi-definite.

This sensitivity result does not directly indicate the signs of the element of $X_\alpha(\alpha^o)$, but this is not possible in a problem as general as ($P^{(i)}$).

Difficulties for sensitivity analysis appear when parameters cater the constraint functions as well.

(ii) Parameters are present not only in the objective but also in the constraint functions and may vary

Let us consider the behavioral system

$$\phi(\alpha) \equiv \max_x \{ f(x,\alpha) \ \big| h(x,\alpha)' = 0'_r \} , \qquad (P^{(ii)})$$

examined by **Silberberg** (1974). Its solution x(α) λ(α) satisfies

$$\rho\ (H_x\ (x(\alpha), \alpha)) = r , \qquad (R_x)$$

$$\text{f-o-c } \{ f_x\ (x(\alpha), \alpha) - H_x(x(\alpha), \alpha)\ \lambda(\alpha) \equiv 0_n\ , h(x(\alpha), \alpha) \equiv 0_r \} \qquad (5)$$

and

$$\text{s-o-c} \left\{ \begin{array}{l} \text{The n x n hessian matrix} \\ F_{xx}(x(\alpha),\alpha) - H_{xx}(x(\alpha), \lambda(\alpha),\alpha) \\ \text{is negative semi} - \text{definite (or definite)} \\ \text{on } T_{x(\alpha)} = \{\eta \in R^n \;\; | H_\lambda(x(\alpha),\alpha)'\eta = 0_r\} \\ (\text{and } \eta \neq 0_n) \end{array} \right\} . \qquad (6)$$

Although we know that the bordered hessian is invertible and, thus, we could derive the fundamental matrix equation for sensitivity analysis, we also know that such a procedure would lead us outside of the Primal method. The difficulty that we face has nothing to do with **Caratheodory**'s theorem. Rather it is due to the simple fact that the hessian of (P^{ii}) has the property of being negative semi-definite (or definite) **only when** it is restricted to its tangent subspace. In fact we know nothing about the n x n hessian, except that its restriction to the tangent subspace, a matrix of dimensions (n – r) x (n – r), must be so. However, before we can say anything about the restricted hessian, we have first to derive it from the hessian and explain the meaning of its being negative semi-definite (or definite). What must be done is known thanks to the painstaking analysis of **Luenberger** (1973), who pointed out the importance of the restricted hessian that is involved in the s-o-c of (P^{ii}).

The restricted hessian can be derived from the hessian, if we consider an n x (n – r) matrix E, with $\rho(E) = n - r$, which satisfies the constraints appearing in s-o-s-c of (P^{ii}).

Indeed, using the theorem of **Lagrange**, if the first n – r choice variables, y, are selected, the rest r choice variables, z(y), are implicitly determined by the r constraints. With x´ now given by (y, z(y)), then the above constraints are given by

$$(H_y(y, z(y), \alpha) \;,\; H_z(y, z(y), \alpha))\, E \;,$$

which leads to r zeros if E is given as

$$E = \begin{bmatrix} I_{(n-r)\,x\,(n-r)} \\ -H_z(y, z(y), \alpha)^{-1}\, H_y(y, z(y), \alpha)' \end{bmatrix},$$ since H_z (y, z(y), α) is an invertible r x r matrix thanks to the Implicit function theorem.

With E′ A E the restricted hessian of A, we get

$$E'AE = [I_{(n-r)\,x\,(n-r)}, -H_y H_z^{-1}] \cdot \begin{bmatrix} F_{yy} - H_{yy}, & F_{yz} - H_{yz} \\ F_{zy} - H_{zy}, & F_{zz} - H_{zz} \end{bmatrix} \begin{bmatrix} I_{(n-r)\,x\,(n-r)} \\ -H_z^{-1} H_y' \end{bmatrix} =$$

$$= \{F_{yy} - H_{yy} - H_y H_z^{-1}[F_{zy} - H_{zy}],\ F_{yz} - H_{yz} - H_y H_z^{-1}[F_{zz} - H_{zz}]\} \cdot \begin{bmatrix} I_{(n-r)\,x\,(n-r)} \\ -H_z^{-1} H_y' \end{bmatrix} =$$

$$= F_{yy} - H_{yy} - H_y H_z^{-1}[F_{zy} - H_{zy}] - [F_{yz} - H_{yz}] H_z^{-1} H_y' + H_y H_z^{-1}[F_{zz} - H_{zz}] H_z^{-1} H_y'.$$

Then, for ε consisting of multiplies of the first n – r elements of each column of E, the quadratic form

$$\begin{aligned} &\varepsilon'[F_{yy} - H_{yy}]\varepsilon - \varepsilon' H_y H_z^{-1}[F_{zy} - H_{zy}]\varepsilon - \varepsilon'[F_{yz} - H_{yz}] H_z^{-1} H_y\, \varepsilon + \\ &+\varepsilon' H_y H_z^{-1} \cdot [F_{zz} - H_{zz}] H_z^{-1} H_y'\, \varepsilon = \varepsilon'[F_{yy} - H_{yy}]\varepsilon < 0 \end{aligned} \qquad (7)$$

and the restricted hessian of A is represented by $F_{yy} - H_{yy}$, which is a symmetric and negative definite (n – r) x (n – r) matrix. These s-o-s-c are the curvature conditions, for any $\varepsilon \in R^{n-r}$, $\varepsilon \neq 0_{n-r}$, which are needed for determining the signs of the rates of change of $Y_\alpha(\alpha)$ with the Primal method.[3]

The above difficulty can be overcome if we consider the less general but simplex behavioral system examined by **Hatta** (1980).

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[3] With x′ denoted by (y, z(y)), $X_\alpha(\alpha) = \begin{bmatrix} Y_\alpha(\alpha) \\ Z_y(y) Y_\alpha(\alpha) \end{bmatrix} - \begin{bmatrix} I_{(n-r)\,x\,(n-r)} \\ Z_y(y) \end{bmatrix} Y_\alpha(\alpha).$

(iii) Constraint levels are non zero and may vary

If (P) takes the form of

$$\phi(\gamma) \equiv \max_{x} \{ f(x) \ \big| h(x)' - \gamma' = 0_r' \} \ , \qquad (P^{iii})$$

then the solution is $x(\gamma)$, $\lambda(\gamma)$. Thus differentiating the f-o-identities w/r to x and $-\lambda$ we get the fundamental matrix equation

$$\begin{bmatrix} X_\gamma(\gamma) \\ -\Lambda_\gamma(\gamma) \end{bmatrix} = \begin{bmatrix} U \ , & V \\ V' , & W \end{bmatrix} \begin{bmatrix} O_{nr} \\ I_{rr} \end{bmatrix} = \begin{bmatrix} V \\ W \end{bmatrix} \qquad (8)$$

which at first sight does not permit the derivation of any sensitivity results about $X_\gamma(\gamma)$.

Fortunately, however, it is possible to express (P^{iii}) in a slightly different manner so as to exploit the specific parameter structure, as **Hatta** did: let us consider for any fixed z , in the domain of definition of h(x), with h (z) = γ, the problem

$$\phi(h(z)) \equiv \max_{x} \{ f(x) \ \big| h(x)' = h(z)' \} \ . \qquad (P_{cv}^{iii})$$

Under this specification each constraint does not respect any fixed level, as in (P^{iii}), but is required to "pass" through some fixed z in R^n. Since z does not vary, the solution $x(h(z)) \neq z$ in general, while

$$\phi(h(z)) \equiv f(x(h(z))) \equiv \max_{x} \{ f(x(h(z)) \} \ . \qquad (P_{cv}^{iii})$$

We thus see that (P_{cv}^{iii}) can be expressed as an unconstrained optimization problem for any such fixed z.

Here, $f(z) \leqq \phi(h(z))$ with

$f(x(\gamma)) \equiv \phi(h(x(\gamma)))$ and so at the local maximum we have

$$\text{f-o-c} \qquad \{ f_x(x(\gamma)) = 0_n \} \qquad (9)$$

and

$$\text{s-o-c} \quad \left\{ \begin{array}{l} F_{xx}(x(\gamma)) \text{ is a negative} \\ \text{semi} - \text{definite (or definite)} \\ \text{for all } \eta \in R^n \text{ (and } \eta \neq 0_n) \end{array} \right\} \quad . \qquad (10)$$

(iv) Parameters appear in all functions as well as constraint levels and may vary

Let us, finally, consider the problem

$$\phi(\alpha,\gamma) \equiv \max_x \{ f(x,\alpha) \ \big| h(x,\alpha)' - \gamma' = 0_r' \} \ , \qquad (P^{iv})$$

with κ additional parameters entering all functions, so that $m = \kappa + r$ and $\kappa > 0$. (P^{iv}) is characterized by (R_x), by

$$\text{f-o-c} \quad \{ f_x(x,\alpha) - H_x(x,\alpha)\lambda = 0_n \ , \ \ h(x,\alpha) - \gamma = 0_r \} \qquad (11)$$

and by

$$\text{s-o-c} \quad \left\{ \begin{array}{l} F_{xx}(x,\alpha) - H_{xx}(x,\lambda,\alpha) \text{ is a negative} \\ \text{semi} - \text{definite (or definite) matrix} \\ \text{on the tangent subspace} \\ T_x = \{ \eta \in R^n \ \big| H_x(x,\alpha)'\eta = 0_r \} \\ (\text{and } \eta \neq 0_n) \end{array} \right\} \qquad (12)$$

If s-o-s-c hold, then the solution $x(\alpha, \gamma), \lambda(\alpha, \gamma) \in C^1$ in (α, γ) near (α^o, γ^o).

As (α, γ) vary around (α^o, γ^o) we could get the fundamental matrix equation

$$\begin{bmatrix} X_\alpha(\alpha^o,\gamma^o) \ , & X_\gamma(\alpha^o,\gamma^o) \\ -\Lambda_\alpha(\alpha^o,\gamma^o) \ , & -\Lambda_\gamma(\alpha^o,\gamma^o) \end{bmatrix} = \begin{bmatrix} U, & V \\ V', & W \end{bmatrix} \begin{bmatrix} -[F^o_{x\alpha} - H^o_{x\alpha}], & O_{nr} \\ -H^{o\prime}_\alpha \ , & I_{rr} \end{bmatrix} = \qquad (13)$$

$$= \begin{bmatrix} -U[F^o_{x\alpha} - H^o_{x\alpha}] - VH^{o\prime}_\alpha \ , & V \\ -V'[F^o_{x\alpha} - H^o_{x\alpha}] - WH^{o\prime}_\alpha \ , & W \end{bmatrix}$$

and examine its implications for the combined parameter variations

and

$$X_\alpha(\alpha^o,\gamma^o) + X_\gamma(\alpha^o,\gamma^o)H_\alpha^{o\prime}$$
$$\Lambda_\alpha(\alpha^o,\gamma^o) + \Lambda_\gamma(\alpha^o,\gamma^o)H_\alpha^{o\prime}\ .$$

Again, however, this procedure leads us outside the Primal method, as in Section 3 (ii) and exactly for the same reasons. Looking back at the s-o-s-c of (P^{iv}), it is the restricted hessian on the tangent subspace that must be negative definite.

The restricted hessian of (P^{iv}) is specified, in Section 3(ii) and is given by the (n – r) x (n – r) matrix F_{yy} (y, z(y), α) – H_{yy}(y, z(y), λ, α).

Thus with the help again of **Luenberger** (1973), we are able to derive curvature conditions that can determine the signs of the rates of change Y_α (α, γ) and Y_γ (α, γ).

4. The Dual method of sensitivity analysis, via the Envelope theorem

This section focuses on the Dual method of sensitivity analysis, in which parameters of the original optimization problem become the choice variables, while the former choice variables are treated as parameters. The Envelope theorem provides a simple and unified formulation of the Dual method, by comparing the value of the objective function at a fixed $x^o = x(\alpha^o)$ to that of the maximal value function of the original problem.

To proceed we must first consider the **Envelope theorem** and examine its first-order Envelope tangencies and second-order envelope curvature conditions.

For the problem examined by **Silberberg** (1974), in section 3(ii), the maximal value function has first-order derivatives,

$$\phi_{\alpha}(\alpha) = f_{\alpha}(x(\alpha),\alpha) - H_{\alpha}(x(\alpha),\alpha)\ \lambda(\alpha)$$

as well as second-order derivatives

$$\Phi_{\alpha\alpha}(\alpha) = \left[[F_{\alpha x} - H_{\alpha x}]\ X_{\alpha} + [F_{\alpha\alpha} - H_{\alpha\alpha}] - H_{\alpha}\Lambda_{\alpha}\right],$$

where function arguments, x (α) and α, have been partially omitted to save space. The former express **Envelope tangencies** while the latter – which are rarely mentioned in the economic literature – involve the symmetric matrix $\Phi_{\alpha\alpha}(\alpha)$ and provide the **envelope curvature conditions** between $\phi(\alpha)$ and $f(x,\alpha)$ at the solution point x(α).

For **Hatta**'s (1980) problem, in section 3(iv), $\phi(\alpha,\gamma)$ has first-order derivatives

$$\phi_{\alpha}(\alpha,\gamma) = f_{\alpha}(x(\alpha,\gamma),\alpha) - H_{\alpha}(x(\alpha,\gamma),\alpha)\ \lambda(\alpha,\gamma),$$
$$\phi_{\gamma}(\alpha,\gamma) = \lambda(\alpha,\gamma)$$

and thus

$$\phi_{\alpha}(\alpha,\gamma) + H_{\alpha}(x(\alpha,\gamma),\alpha)\ \lambda(\alpha,\gamma) = f_{\alpha}(x(\alpha,\gamma),\alpha),$$

as well as second-order derivatives

$$\Phi(\alpha,\gamma) \equiv \begin{bmatrix} \Phi_{\alpha\alpha}(\alpha,\gamma), & \Phi_{\alpha\gamma}(\alpha,\gamma) \\ \Phi_{\gamma\alpha}(\alpha,\gamma), & \Phi_{\gamma\gamma}(\alpha,\gamma) \end{bmatrix} =$$
$$= \begin{bmatrix} [F_{\alpha x} - H_{\alpha x}]X_{\alpha} + [F_{\alpha\alpha} - H_{\alpha\alpha}] - H_{\alpha}\Lambda_{\alpha} & , & [F_{\alpha x} - H_{\alpha x}]X_{\gamma} - H_{\alpha}\Lambda_{\gamma} \\ \Lambda_{\alpha}(\alpha,\gamma) & , & \Lambda_{\gamma}(\alpha,\gamma) \end{bmatrix}.$$

Again the former express **envelope tangencies**, while the latter involve the symmetric matrix $\Phi(\alpha,\gamma)$ and provide **envelope curvature conditions** between $\phi(\alpha,\gamma)$ and f(x, α) at the solution point x(α, γ).

As in Section 3 we consider several cases

(i) Parameters appear only in the objective function

If the original optimization problem is

$$\phi(\alpha) \equiv \max_{x} \{ f(x,\alpha) \ \big| h(x)' = 0_r' \} \ , \qquad (P^{(i)})$$

as $x^o = x(\alpha^o)$, $\lambda^o = \lambda(\alpha^o)$, we may examine the dual method of sensitivity analysis via the Envelope problem

$$0 \equiv \max_{\alpha} \{ f(x^o,\alpha) - \phi(\alpha) \ \big| h(x^o)' = 0_r' \} \ ,$$

or ,

$$0 \equiv \max_{\alpha} \{ f(x^o,\alpha) - \phi(\alpha) \} \ , \qquad (D^i)$$

since x^o remains constant as a vary and, thus, feasibility is preserved.
The solution of (D^i) satisfies

$$\text{f-o-c} \quad \{ f_\alpha(x^o,\alpha^o) - \phi_\alpha(\alpha^o) = 0_m \} \qquad (14)$$

and

$$\text{s-o-c} \quad \left\{ \begin{array}{l} \text{The m x m matrix } F_{\alpha\alpha}(x^o,\alpha^o) - \Phi_{\alpha\alpha}(\alpha^o) = \\ = F^o_{\alpha\alpha} - F^o_{\alpha x} X_\alpha(\alpha^o) - F^o_{\alpha\alpha} = -F^o_{\alpha x} X_\alpha(\alpha^o) \\ \text{is negative semi - definite (or definite)} \end{array} \right\} \qquad (15)$$

Thus again $F^o_{\alpha x} X_\alpha(\alpha^o)$ is a symmetric and positive semi-definite matrix.

(ii) Parameters appear in all functions of the original optimization problem

If we consider the problem

$$\phi(\alpha) \equiv \max_{x} \{ f(x,\alpha) \ \big| h(x,\alpha)' = 0_r' \} \ , \qquad (P^{ii})$$

examined by **Silberberg** (1974), with $x(\alpha)$, $\lambda(\alpha)$ its solution, then letting $x^o = x(\alpha^o)$ and $\lambda^o = \lambda(\alpha^o)$ we can examine the Dual method via a constrained Envelope problem

$$0 \equiv \max_{\alpha} \{ f(x^o, \alpha) - \phi(\alpha) \ \big| h(x^o, \alpha)' = 0_r' \}, \qquad (D^{ii})$$

(D^{ii}) is a constrained problem, since as a vary feasibility is not preserved, even if x^o is constant. We also note that (D^{ii}) is well defined when $m > r \geq 1$ and

$$\rho\,(H_\alpha(x^o, \alpha)) = r \qquad (R_\alpha)$$

Then we have

$$\text{f-o-c} \quad \{ f_\alpha(x^o, \alpha) - \phi_\alpha(\alpha) - H_\alpha(x^o, \alpha)\xi = 0_m,\ h(x^o, \alpha) = 0_r \}, \qquad (16)$$

which are satisified at least at α^o and $\xi^o = \lambda^o$

and

$$\text{s-o-c} \quad \left\{ \begin{array}{l} \text{The m x m matrix } F^o_{\alpha\alpha} - \Phi^o_{\alpha\alpha} - H^o_{\alpha\alpha} = \\ = -[F^o_{\alpha x} - H^o_{\alpha x}]X_\alpha(\alpha^o) - H^o_\alpha \Lambda_\alpha(\alpha^o) = \\ = -[F^o_{\alpha x} - H^o_{\alpha x}]X_\alpha(\alpha^o) \\ \text{is a negative semi} - \text{definite (or definite)} \\ \text{on the tangent subspace } T_\alpha = \{\zeta \in R^m \ \big| \\ \big| H^{o\prime}_\alpha \zeta = 0_r\} \text{ (and } \zeta \neq 0_m) \end{array} \right\} \qquad (17)$$

The f-o-c and s-o-c are given in **Silberberg** (1974) in equations (6) and (10), respectively, where it is also noted that (10) are subject to constraints.

It is again obvious that the above s-o-c cannot be used directly for sensitivity analysis. Indeed, we know nothing about the m x m matrix, $-[F^o_{\alpha x} - H^o_{\alpha x}]X_\alpha(\alpha^o)$, except that its restriction in the tangent subspace must be negative semi-definite (or definite).

Again using the m x (m – r) matrix $E^o = \begin{bmatrix} I_{(m-r)\,x\,(m-r)} \\ -H_\gamma^{o-1} H_\beta^{o'} \end{bmatrix}$, which provides a basis for all $\varepsilon \in R^{m-r}$ consisting of multiples of the first m – r elements of each column of E^o, since $(H_\beta^o, H_\gamma^o)E^o = (H_\beta^{o'} - H_\beta^o) = 0_r'$.

We may derive the restricted (m – r) x (m – r) matrix

$$-E^{o'} \cdot \begin{bmatrix} F_{\beta x}^o - H_{\beta x}^o \\ F_{\gamma x}^o - H_{\gamma x}^o \end{bmatrix} [X_\beta^o, X_\gamma^o] E^o = [-[F_{\beta x}^o - H_{\beta x}^o] + \\ + H_\beta^o H_\gamma^{o^{-1}} [F_{\gamma x}^o - H_{\gamma x}^o]][X_\beta^o - X_\gamma^o H_\gamma^{o^{-1}} H_\beta^{o'}] \quad . \tag{18}$$

Then we get the Envelope curvature conditions

$$e - c - c\left\{\begin{matrix} \text{The (m – r) x (m – r) matrix in (18)} \\ \text{is negative semi – definite (or definite)} \end{matrix}\right\} \tag{19}$$

As for the second matrix approving in (18), it has a clear meaning for us economists: it is the matrix of compensated variations in parameters β, as it shows the rates of change of x(β, γ) in β, when the remaining parameters, γ, also vary appropriately so as to keep the constraints h (x^o, β, γ) = 0_r satisfied. We also know that the nullity of this matrix is at least equal to r. Indeed, differentiating h(x (β, γ), β, γ) = 0_r w/r to β and γ we get $H_x' X_\beta + H_\beta' = O_{r(m-r)}$ and $H_x' X_\gamma + H_\gamma = O_{rr}$. From the latter we get $H_x' X_\gamma - H_\gamma^{-1} H_\beta' + H_\beta' = O_{r(m-r)}$, which when subtracted from the first gives us

$$H_x' [X_\beta - X_\gamma H_\gamma^{-1} H_\beta'] = O_{r(m-r)} \quad .$$

(iii) Parameters appear in all functions and also as constraint levels

Let us consider the **Hatta** (1980) model, i.e., problem

$$\phi(\alpha, \gamma) \equiv \max_x \{ f(x, \alpha) \;\big|\; h(x, \alpha)' - \gamma' = 0_r' \} \;, \tag{P^{iv}}$$

with m parameters, $m = \kappa + r$, the first κ of which appear in the objective and constraint functions. This is less general than **Silberberg** (1974), mainly because the r constraint functions $h(x, \alpha)' - \gamma' = 0'_r$ have a more specific structure in γ. But we will see again that this simplicity is more than compensated by the additional results obtained. The solution of (P^{iv}), $x(\alpha, \gamma)$ and $\lambda(\alpha, \gamma)$, attains a (strict) local maximum (if s-o-s-c hold).

The maximal value function $\phi(\alpha,\gamma) \equiv f(x(\alpha,\gamma),\alpha)$

has first-order derivative properties

$$\phi_\alpha(\alpha,\gamma) = f_\alpha(x(\alpha,\gamma),\alpha) - H_\alpha(x(\alpha,\gamma),\alpha)\lambda(\alpha,\gamma)$$

and $$\phi_\gamma(\alpha,\gamma) = \lambda(\alpha,\gamma)$$

and second-order derivative properties, which involve the symmetric matrix

$$\Phi(\alpha,\gamma) \equiv \begin{bmatrix} \Phi_{\alpha\alpha}, & \Phi_{\alpha\gamma} \\ \Phi_{\gamma\alpha}, & \Phi_{\gamma\gamma} \end{bmatrix} = \begin{bmatrix} [F_{\alpha x} - H_{\alpha x}]X_\alpha + F_{\alpha\alpha} - H_{\alpha\alpha} - H_\alpha\Lambda_\alpha & , & [F_{\alpha x} - H_{\alpha x}]X_\gamma - H_\alpha\Lambda_\gamma \\ \Lambda_\alpha & , & \Lambda_\gamma \end{bmatrix}, \quad (20)$$

For any (α^o, γ^o) let $x^o = x(\alpha^o, \gamma^o)$ and $\lambda^o = \lambda(\alpha^o, \gamma^o)$ and examine either a constrained Envelope problem

$$0 \equiv \max_{\alpha,\gamma} \{ f(x^o,\alpha) - \phi(\alpha,\gamma) \mid h(x^o,\alpha)' - \gamma' = 0'_r \} \qquad (D_c^{iv})$$

with m (α, γ), $m > r$, and (R_α) holding,

or an unconstrained Envelope problem

$$0 \equiv \max_{\alpha} \{ f(x^o,\alpha) - \phi(\alpha, h(x^o,\alpha)) \}, \qquad (D^{iv})$$

with

$$\text{f-o-c} \left\{ f_\alpha(x^o,\alpha) - \phi_\alpha(\alpha, h(x^o,\alpha) - H_\alpha(x^o,\alpha)\phi_\gamma(\alpha, h(x^o,\alpha) = 0_\kappa \right\} \quad (21)$$

which are satisfied at least at α^o, $h(x^o, \alpha^o) = \gamma^o$, as well as with

$$\text{s-o-c} \left\{ \begin{array}{l} \text{The } \kappa \text{ x } \kappa \text{ matrix } F^o_{\alpha\alpha} - \Phi^o_{\alpha\alpha} - \Phi^o_{\alpha\gamma}H^o_\alpha - \\ -H^{o\prime}_\alpha\Phi^o_{\gamma\alpha} - H^{o\prime}_\alpha\Phi^o_{\gamma\gamma}H^o_\alpha - H_{\alpha\alpha}(x^o, \phi^o_\gamma, \alpha^o) \\ \text{is negative semi-definite (or definite)} \end{array} \right\} \qquad (22)$$

At α^o a (strict) local maximum of zero is attained (if s-o-s-c hold). Using then the second-order derivatives in (20), we can easily show the envelope curvature conditions of (D^{iv})

$$\text{e-c-c} \begin{Bmatrix} \text{The } \kappa \text{ x } \kappa \text{ matrix } -[F^o_{\alpha x} - H^o_{\alpha x}][X^o_\alpha + X^o_\gamma H^{o\prime}_\alpha] \\ \text{is negative semi} - \text{definite (or definite)} \end{Bmatrix} . \qquad (23)$$

The reader can check that the same e – c - c are derived via (D^{iv}_c), as well as that the e – c - c in the **Silberberg** model of 4(ii) are reduced to (23) when $h(x, \alpha, \gamma) = 0_r$ becomes $h(x, \alpha) - \gamma = 0_r$.

Finally, the absence in the dual method of sensitivity analysis, of the simple case of **Hatta**'s (1980) that appeared in section 3(iii), is due to the fact that the number of parameters in this case is only equal to that of constraints. Thus the dual problem via the Envelope theorem is not well defined.

5. Concluding Remarks

Since the late 1930's and the early 1940's, economists like **Hicks** (1939) and, especially, **Samuelson** (1947), started a systematic analysis of comparative static results about the signs of the rates of change of choice variables as parameters of the system vary. Indeed when the behavioral system is expressed as an optimization problem such results on sensitivity analysis of their solutions, as parameters vary, became readily available. **Samuelson** (1947) developed the Primal method of sensitivity analysis through the use of second-order-conditions for an optimum, which specify that the hessian of an unconstrained optimization problem must be semi-definite or definite, while in constrained problems the above hessian must be so **only when** it is restricted on its tangent subspace. **Samuelson** also introduces the Envelope theorem into economic theory.

However, his emphasis on the first-order envelope tangency between the envelope curve or surface and the curves or surfaces that it touches, did not permit him to develop the Dual method of sensitivity analysis.

The **Dual method** was first examined by **Silberberg** (1974), who considered a general constrained optimization problem with m parameters appearing in all functions. If $m > r \geq 1$ and the gradient of $h(x, \alpha)$ in α, $H_\alpha\, x, \alpha)$, has $\rho\,(H_\alpha\,(x, \alpha) = r$, then **Silberberg's** dual problem is given by

$$0 \equiv \max_{\alpha}\{d(\alpha; x)\} \equiv \max_{\alpha}\{f(x,\alpha) - \phi(\alpha) \;\big|\, h(x,\alpha)' = 0'_r\;, \qquad (D_s)$$

which is non other than our Dual problem, via the Envelope theorem.

We must note that **Silberberg** thought, wrongly, that he could develop a "Primal-Dual" method of sensitivity analysis, but he showed that this was not possible. He also noted that the s-o-c of (D_s) are subject to constraints, but he did not attempt to produce curvature conditions suitable for sensitivity analysis.

Reference must also be made to **Pauwels'** (1979) paper, as well as to two more recent papers by **Partovi and Caputo** (2006), (2007), which are quite relevant for us and deserve more attention than can be paid here.

Pauwels' purpose was "to show that Silberberg's main result can also easily be derived without introducing the Primal-Dual problem" and also to sharpen and correct some of the results. **Pauwels**, however, was not clear about the distinction between the Primal and Dual methods; otherwise he should have realized that the linear subspace $V \in R^m$ defined "by $V = \{u \in R^m \;\big|\, g_\alpha u = 0\}$" is non other than the tangent subspace that appears in the s-o-c of (D^{ii}) in equation (16), above, and not the tangent subspace that appears in the s-o-c of

(P^{ii}) in (6). Thus **Pauwels** uses the dual method of sensitivity analysis to derive **Silberberg's** main result in his equation (10).[4]

Partovi and Caputo examine general procedures for deriving the restricted Hessian that may appear in the s-o-c of constrained optimization problems, from the original Hessian of the problem. Since they deal mainly with **Silberberg** (1974) and **Hatta** (1980), they consider the dual method of comparative statics. However, the road that they follow can easily become quite treacherous, due to the existence of alternative m x (m - r) matrices E, one of which is perhaps the most appropriate one, because it leads to an easily interpretable explanation of the meaning of a semi-definite or definite restricted Hessian. Indeed, their general procedure for using projection matrices leads them, on the one hand, to matrices that are negative semi-definite or definite without any constraint but, on the other hand, these matrices bear no relation to the restricted Hessians that are involved. Indeed, even in the (2006) paper, where they start with concepts like compensated partial derivatives, they do not derive anything like the **Slutsky** matrix for the **Silberberg** model but only for the simpler **Hatta** model.

The advantage of **Hatta's** (1980) "gain function" method, $g(\alpha, z) \equiv \phi(\alpha, h(\alpha, z) - f(\alpha, z)$ for any fixed $z \in X \subset R^n$, is that it leads directly to curvature conditions suitable for sensitivity analysis. Indeed the negative of g (α, z) is non other but our (D^{iv}) for $z = x^o = x(\alpha^o)$.

Using matrix theory terms, both the Primal and Dual methods of sensitivity analysis are elegant and relatively simple and quite transparent. When

[4] We also note that **Pauwels**, too, like **Silberberg** and so many economists, does not appear to comprehend the drawback of s-o-c in (16) above and the importance of deriving curvature conditions on the (m – r) x (m – r) restricted matrix in equation (17), above.

parameters enter the constraint functions, sensitivity results are produced only for a subset of choice variables, $n - r$ in the primal and $m - r$ in the dual method. Indeed, both methods require that $n > r \geq 1$ and $m > r \geq 1$, respectively.[5]

Finally, the importance of **Luenberger's** (1973) contribution is more than obvious. In his chapter on constrained optimization problems, **Luenberge**r considers first the regularity assumption, under which the tangent subspace can be expressed in terms of the gradient matrix of the constraints, which is a subspace of dimensions $n - r$ of $X \subset R^n$. He then offers a painstaking analysis of the distinction between the Hessian of the problem and the restricted Hessian that appears in the second-order-conditions. Since the s-o-c are subject to constraints, it is the latter Hessian that must be semi-definite, or definite. The restricted Hessian is of dimensions $(n - r) \times (n - r)$ with the primal method and must be derived from the original $n \times n$ Hessian, so that the meaning of the above properties can become transparent.

Again with the dual method, **Luenberger's** methodology can be used, equally well, so as to help in deriving the $(m - r) \times (m - r)$ restricted Hessian from the $m \times m$ Hessian of the dual problem.

It must be noted, however, that the derivation of the reduced Hessian from the original one can be done in several alternative ways. Thus depending on the particular objectives of each research study, there is an E matrix that leads to E′

[5] We may also note that the Envelope theorem, used in the Dual methods can be applied quite profitably in producing additional manifestations of the **Le Chatelier Principle**. See e.g. **Drandakis** (2009).

to E′ A E , or to (18) in section 4, and a reduced restricted Hessian with the best possible interpretation of its semidefiniteness or definiteness.[6, 7]

In conclusion, it must be freely admitted that, although economists were able to develop quickly comparative static results for optimization problems dealing with consumer choice and the theory of the firm, they were rather slow in their attempts to do so for more abstract general optimization problems. Indeed, we owe a lot to the contribution of both "econometricians", like **Barten**, **Dhrymes** and others, who introduced matrix theory methods into sensitivity analysis and to mathematicians, like **Caratheodory** and **Luenberger**, without the help of which neither the Primal nor the Dual methods would have become completely operational.

[6] Indeed, the lack of attention being paid in the economic literature to **Luenberger's** (1973) contribution is probable due to the unfortunate fact that his interests were not focused on sensitivity analysis. Thus **Luenberger's** E has no relation to our own E. Our E is simple and quite natural, given our interest in sensitivity analysis and our exposure to the usefulness of "compensated parameter variations".

[7] That our E is the appropriate one for our purpose is verified in **Drandakis** (2010), where Profit maximization under "point" and "straight" rationing is examined, once more. Due to the simplicity of unrestricted profit, restrict profit maxima can be given both as unconstrained and constrained problems. It is easily checked that an E converts the constrained problems into the unconstrained ones.

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